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ON CONVERGENCE OF ITERATIVE METHODS IN PLASTIC STRAIN ANALYSIS

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Abstract—Two general formulations and solution techniques for plane strain plasticity problems are investigated: the effective force and material-stiffness matrix iteration methods. Compatible finite-difference models are derived by minimizing a discretization of the potential energy function over a general boundary configuration, using the Hencky–Nadai hardening law to mathematically model the material behavior. The resultant material stiffness matrices are symmetric, positive definite, and block tri-diagonal. Analytical and numerical studies of the accuracy and convergence characteristics of the methods are given for matrices of order 1000.

1. INTRODUCTION

THE well known total laws of the deformation theory of plasticity constitute a valuable model for thermomechanical plastic strain analysis. Their utility in problems of non-cyclic loading is firmly established through extensive application, and theoretical objections to their use have been clearly answered in the literature [1–6].

One of the more widely used deformation theories is that of the Hencky–Nadai hardening law based upon the von Mises yield criterion. In numerical stress analyses using this mathematical model, a procedure frequently is followed of formulating the problem as a set of quasi-linear differential equations and then obtaining the solution by an iterative technique. For two and three dimensional problems, this normally involves a finite-difference or finite element discretization, large order matrix equations, and the accompanying questions of consistency of formulation, order of discretization errors, stability of solution algorithm, and, most importantly, the numerical accuracy and efficiency of the overall method. It is the purpose of this paper to analyze and compare, within the framework of these questions, two energy-based finite-difference formulations and solution techniques for elastic–plastic plane strain problems: (1) the material-stiffness matrix iteration method previously suggested by the writer [7], and (2) a generalization of the effective force method of Ilyushin–Mendelson–Manson [8, 9].

2. THE QUASI-LINEAR FIELD EQUATIONS

For a material subjected to both thermal and mechanical loading and obeying the Hencky–Nadai hardening law in the plastic range, the stress–strain law is (using the indicial notation of cartesian tensor calculus)

$$\varepsilon_{ij} = \frac{1}{2G}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} - \alpha T\delta_{ij} + \varepsilon_{ij}^{p} \tag{1}$$

wherein α and T are the coefficient of thermal expansion and the temperature rise, respectively, δ_{ij} is the Kronecker delta, and

$$\varepsilon_{ij}^p = \frac{3}{2}\frac{\bar{\varepsilon}^p}{\bar{\sigma}} s_{ij}. \tag{2}$$

The deviatoric stress tensor $s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$, and the equivalent stress and equivalent plastic strain are related through the tensile test and defined as

$$\bar{\sigma} = \sqrt{(\tfrac{3}{2}s_{kl}s_{kl})}, \qquad \bar{\varepsilon}^p = \bar{\varepsilon}^p(\bar{\sigma}) = \sqrt{(\tfrac{2}{3}\varepsilon_{kl}^p\varepsilon_{kl}^p)}. \tag{3}$$

Combining these equations with the equilibrium equations and the small strain–displacement relations

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{4}$$

Ilyushin [8] derived a set of quasi-linear differential equations by "lumping" all of the plasticity effects into a fictitious body force term. Ilyushin's equations, extended to include thermal loading, can be written

$$(\lambda + G)e_{,i} + G\nabla^2 u_i + F_i - \{(3\lambda + 2G)\alpha T\}_{,i} - R_i = 0. \tag{5}$$

R_i is the effective body force due to plastic strains, given as

$$R_i = 2G(\omega e_{ij})_{,j} \equiv \tfrac{1}{3}G\omega e_{,i} + G\omega\nabla^2 u_i + G\omega_{,j}(u_{i,j} + u_{j,i}) - \tfrac{2}{3}G\omega_{,i}e. \tag{6}$$

The scalar e is the volumetric strain $u_{k,k}$; $e_{ij} = \varepsilon_{ij} - \frac{1}{3}e\delta_{ij}$ is the deviatoric strain tensor; and ω is a plasticity parameter, first defined by Ilyushin, which equals

$$\omega = 1 - \frac{G_{\text{sec}}}{G} = 3G_{\text{sec}}\frac{\bar{\varepsilon}^p}{\bar{\sigma}} \tag{7}$$

where G_{sec} is the secant modulus of the theoretical stress–strain curve of pure shear predicted by the tensile test. The concept of the effective body force has been used in integral equation solutions of several one dimensional problems by Mendelson and Manson [9]. For plane stress and plane strain problems, Ilyushin's concept has been incorporated within a stress function formulation by Roberts and Mendelson [10] and Tuba [11, 12]. More recently, Tuba [13] has given a direct finite-difference displacement formulation of the effective force method and has presented results for the problem of a circular hole in a uniaxially loaded plate. Other applications of the effective body force, also referred to as the initial strain method, have been given by Webster and Ellison [14] and Khojasteh-Bakht and Popov [15].

An alternate set of quasi-linear equations has been derived by the writer [7] in terms of pseudo material properties, using the relative minimum energy principle of Kachanov [16] and Greenberg [17] for the Hencky–Nadai material. These equations are

$$(\lambda^* + G^*)e_{,i} + G^*\nabla^2 u_i + G^*_{,j}(u_{i,j} + u_{j,i}) + \lambda^*_{,i}e + F_i - \{(3\lambda + 2G)\alpha T\}_{,i} = 0. \tag{8}$$

The variable "material properties," λ^* and $G^* = G_{\text{sec}}$, are analogous to the Lamé constants but are defined in terms of the secant modulus $E^* = E_{\text{sec}}$ (from the tensile test) and an effective Poisson's ratio ν^*, given as functions of a plastic strain parameter ψ:

$$E^* = E/(1 + \psi), \qquad \nu^* = (\nu + 0{\cdot}5\psi)/(1 + \psi) \tag{9}$$

with

$$\psi = \bar{\varepsilon}^p/\bar{\varepsilon}^E = E\bar{\varepsilon}^p/\bar{\sigma}. \tag{10}$$

It is easily shown that equations (5) and (8) are algebraically equivalent for a given plastic strain field in a thermo-mechanical deformation problem. Unfortunately, the plastic strains are never "given," and in many cases one begins with an over-stressed elastic solution by taking the plastic strains to be zero. Thus, when these equations are viewed as bases for methods of successive approximation, we see that they embody distinctly different iteration procedures. And if proper care is not exercised in the manner of discretization, they even can differ significantly in their respective numerical models. To avoid this latter difficulty, the technique of forming the quasi-linear difference equations from the discretized potential energy function, given in [7] for the stiffness matrix iteration method [equations (8)], will be extended in Section 3 to the discretization of equations (5) of the effective force method.

3. ENERGY FORMULATION OF DIFFERENCE EQUATIONS

The total potential energy function for the Hencky–Nadai material can be expressed

$$\begin{aligned}\Pi = \frac{1}{2}\int_R [&G(u_{i,j}+u_{j,i})u_{i,j}+\lambda u_{k,k}^2 - 2(3\lambda+2G)\alpha T u_{k,k} \\ &+ 3(3\lambda+2G)(\alpha T)^2 - 2F_i u_i]\,\mathrm{d}R - \int_{S_T} X_i u_i\,\mathrm{d}S \\ &- \int_R \left[G\int_0^{e_{ij}} \omega\,\mathrm{d}(e_{ij}e_{ij})\right]\mathrm{d}R\end{aligned} \tag{11}$$

where S_T is that portion of the boundary S on which tractions are prescribed. The first two integrals correspond to the uncoupled thermoelasticity problem; the third integral is the source of the fictitious body force in Ilyushin's equations (5) [which, of course, can be derived by applying the extremum principle of Kachanov–Greenberg to equation (11)]. For problems of plane strain, replacing x_1, x_2 by x, y, u_1, u_2 by u, v and X_1, X_2 by X, Y, a quadrature formula for the potential energy may be written

$$\begin{aligned}\bar{\Pi} = \bar{\Omega} - \sum_{\bar{R}} &[(3\lambda+2G)\alpha T(\bar{u}_{,x}+\bar{v}_{,y}) + F_x u + F_y v \\ &- \tfrac{3}{2}(3\lambda+2G)(\alpha T)^2]_q \Delta R_q - \sum_{\bar{S}_T}(Xu+Yv)_{q'}\Delta S_{q'} - \sum_{\bar{R}}[G\omega e_{kl}e_{kl}]_q \Delta R_q\end{aligned} \tag{12}$$

with

$$\bar{\Omega} = \sum_{\bar{R}} [G\{\bar{u}_{,x}^2 + \tfrac{1}{2}(\bar{u}_{,y}+\bar{v}_{,x})^2 + \bar{v}_{,y}^2\} + \tfrac{1}{2}\lambda(\bar{u}_{,x}+\bar{v}_{,y})^2]_q \Delta R_q. \tag{13}$$

In equations (12) and (13) the integrations over R and S_T have been replaced by summations over a discrete set of points q and q' corresponding to some finite-difference network, with ΔR_q and $\Delta S_{q'}$ representing the associated incremental areas and boundary curve lengths, respectively. (The horizontal bar above a function indicates the corresponding

discretized form.) The plastic strain term of equation (12) becomes, upon expansion,

$$\sum_{\bar{R}} [G\omega \bar{e}_{kl}\bar{e}_{kl}]_q \Delta R_q = \sum_{\bar{R}} [G\omega\{\tfrac{2}{3}(\bar{u}_{,x}^2 - \bar{u}_{,x}\bar{v}_{,y} + \bar{v}_{,y}^2) + \tfrac{1}{2}(\bar{u}_{,y} + \bar{v}_{,x})^2\}]_q \Delta R_q. \tag{14}$$

The discrete analogy of the continuum minimum energy principle is realized by equating to zero the rate of change of $\bar{\Pi}$ with respect to each unknown displacement. Expressing $\bar{\Pi}$ in matrix terms we have

$$\bar{\Pi} = \tfrac{1}{2}\{\delta\}^T[A]\{\delta\} - \{\delta\}^T\{\delta_0\} + D_T - \tfrac{1}{2}\{\delta\}^T[P]\{\delta\} \tag{15}$$

and the matrix equilibrium equation becomes

$$\left\{\frac{\partial \bar{\Pi}}{\partial \delta}\right\} = [A]\{\delta\} - \{\delta_0\} - [P]\{\delta\} = 0. \tag{16}$$

The symmetric matrix $[A]$ is the coefficients or material-stiffness matrix of the thermoelastic problem. It is of order (and rank) N equal to the number of unknown displacements in $\bar{R} + \bar{S}_T$. $\{\delta\}$ is the column vector of these unknown displacements, and $\{\delta_0\}$ is the "loads" vector for the thermoelastic problem determined from the first temperature term in equation (12), the body force and prescribed traction terms, and prescribed displacements on $\bar{S}_D$. The scalar D_T arises from the second temperature term and terms quadratic in the prescribed displacements. The matrix $[P]$, also of order (but not necessarily rank) N, will be called the plasticity matrix. The quadratic form involving $[P]$ in equation (15) is the sum of all terms in equation (14) which are quadratic in the unknown displacements in $\bar{R} + \bar{S}_T$. For the first boundary value problem of a body in equilibrium under thermal gradients, body forces, and prescribed surface tractions, this sum encompasses all of the terms in equation (15). Thus we can write

$$\tfrac{1}{2}\{\delta\}^T[P]\{\delta\} = \sum_{\bar{R}} [G\omega\{\tfrac{2}{3}(\bar{u}_{,x}^2 - \bar{u}_{,x}\bar{v}_{,y} + \bar{v}_{,y}^2) + \tfrac{1}{2}(\bar{u}_{,y} + \bar{v}_{,x})^2\}]_q \Delta R_q. \tag{17}$$

From equation (7), $G\omega \geq 0$ for all strain levels in a Hencky–Nadai material with a monotonically increasing stress–strain curve. The sum of the terms within the braces is easily shown to be positive semi-definite; hence, equation (17) represents a positive semi-definite quadratic form and $[P]$ is a symmetric, positive semi-definite (or non-negative definite) matrix. $[P]$ clearly is not positive definite in general, for in problems with regions of purely elastic response the rank of $[P]$ will be less than the order N. If the entire body is in the elastic range, $[P]$ is of course the null matrix. Consider now the symmetric matrix $[A]$. For the first boundary value problem defined above, the positive semi-definite quadratic form $\bar{\Omega}$ is

$$\bar{\Omega} = \tfrac{1}{2}\{\delta\}^T[A]\{\delta\} \tag{18}$$

from which $[A]$ is a positive semi-definite matrix. Since $[A]$ corresponds to the elastic problem $[A]\{\delta\} = \{\delta_0\}$, it should yield finite-difference equations which are consistent in the sense defined by Tanrikulu and Prager [18]. This can be accomplished (to a degree) for a network consisting of rectangular and triangular grid elements over an arbitrary domain, as shown in Fig. 1, by using quadrature formulas of the type suggested in [7]. Appropriate formulas for evaluation of the terms in equation (13) are given in the Appendix, together with similar formulas for equation (14). From the comparison therein of the resulting difference equation in the X-direction with the corresponding equation from the

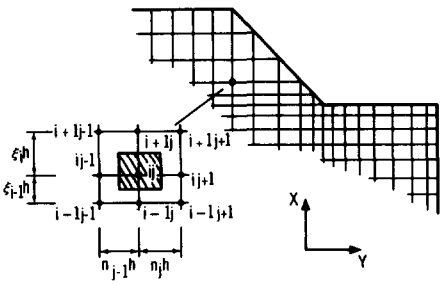

FIG. 1. Interior point of finite-difference grid.

matrix iteration method (derived in [7]), it is seen that the discrete models of the alternate quasi-linear formulations of the plane strain plasticity problem are compatible.

4. ANALYTICAL COMPARISON OF SUCCESSIVE SOLUTION METHODS

The recursion formulas for the nth iteration step according to the equivalent force method of equation (16) are

$$\begin{aligned} [A]\{\delta_n\} &= \{\delta_0\} + [P_n]\{\delta_{n-1}\}, \qquad n > 1 \\ [P_n] &= [P(\omega_n(x, y))], \qquad \omega_n(x, y) = \omega(\delta_{n-1}) \end{aligned} \tag{19}$$

with the attendant starting condition

$$[A]\{\delta_1\} = \{\delta_0\} \tag{20}$$

such that $[P_1]$ is the null matrix.

Analogous equations for the nth iteration of the material-stiffness matrix as given in [7] are

$$\begin{aligned} [A_n^*]\{\delta_n\} &= \{\delta_0\}, \qquad n > 1 \\ [A_n^*] &= [A^*(\psi_n(x, y))], \qquad \psi_n(x, y) = \psi(\delta_{n-1}) \end{aligned} \tag{21}$$

with a starting condition identical to equation (20). The matrix A^*, corresponding to finite-difference equations of the type equation (A6), is positive definite. Its elements are bounded for any set of λ^*, G^* physically obtainable from an elastic Poisson's ratio in the range $0 < \nu < 0{\cdot}5$. Consequently, the iteration as defined by equations (21) cannot diverge unbounded [7].

A sufficient condition for convergence of this method of successive solutions is that ψ increase monotonically from zero over all network points experiencing plastic deformation, a proof of which is given in [7]. It can be argued that monotonic increase in ψ (hence ω) is also a sufficient condition for (1) monotonic convergence to the discretized minimum potential energy function from above and (2) convergence by the iterative method of equations (21) in fewer number of cycles than by the method of equations (19). Consider the initial approximation to $\bar{\Pi}$ which corresponds to the "elastic" solution of equation (20). This first approximation can be written

$$\bar{\Pi}_1 = -\tfrac{1}{2}\{\delta_1\}^T\{\delta_0\} + D_T. \tag{22}$$

An improved value is obtained by subtracting the quadratic term $\frac{1}{2}\{\delta_1\}^T[P_2]\{\delta_1\}$ computed from the first solution vector $\{\delta_1\}$. Although $[P_2]$ is only a positive semi-definite matrix, the quadratic form will be positive since $\{\delta_1\}$ is a non-zero displacement field over all network points with which non-zero values in $[P_2]$ are associated (unless $[P_2]$ is the null matrix, in which case the elastic solution is the final solution). Hence, the first correction to the energy is algebraically less than the initial value $\bar{\Pi}_1$. Consider now the discretized energy corresponding to the nth successive solution of equations (21):

$$\bar{\Pi}_n = \tfrac{1}{2}\{\delta_n\}^T[A]\{\delta_n\} - \{\delta_n\}^T\{\delta_0\} + D_T - \tfrac{1}{2}\{\delta_n\}^T[P_n]\{\delta_n\} \tag{23}$$

where

$$[A] - [P_n(\delta_{n-1})] = [A_n^*(\delta_{n-1})]. \tag{24}$$

If we were to end the iterative process at this point, a final improved value for $\bar{\Pi}$ could be written

$$\bar{\Pi}'_n = \tfrac{1}{2}\{\delta_n\}^T[A]\{\delta_n\} - \{\delta_n\}^T\{\delta_0\} + D_T - \tfrac{1}{2}\{\delta_n\}^T[P_{n+1}]\{\delta_n\} \tag{25}$$

with $[P_{n+1}] = [P(\delta_n)]$. From equations (7), (9), (10), (14) and (17), the plastic strain term in the potential energy function may be expressed

$$\tfrac{1}{2}\{\delta\}^T[P]\{\delta\} = \sum_{\bar{R}} [G\omega\bar{e}_{kl}\bar{e}_{kl}]_q \Delta R_q = \sum_{\bar{R}} \tfrac{2}{3}[G\omega(1+\nu^*)^2\bar{\varepsilon}^2]_q \Delta R_q. \tag{26}$$

From equation (9), it is seen that ν^* increases with ψ. The total equivalent strain

$$\bar{\varepsilon} = \bar{\varepsilon}_p + \bar{\sigma}/E = (1+\psi)\bar{\sigma}/E \tag{27}$$

at each point q also increases with ψ. Alternately, it can be shown that the elements in $[P]$ increase linearly with the function

$$\beta_n = \frac{\Delta\psi_n}{(1+\nu+1{\cdot}5\psi_n)(1+\nu+1{\cdot}5\psi_{n-1})}. \tag{28}$$

Thus, the quadratic form of equation (26) increases monotonically for monotonic increase in ψ over all (or a sufficient number) of the network points, and $\bar{\Pi}'_n < \bar{\Pi}_n$. Extending this argument, we conclude that the potential energy will converge to the minimum value from above monotonically with ψ. Numerical demonstration of such convergence is given in Section 6.

To compare the relative convergence rates of the matrix iteration and effective force (vector iteration) methods, consider the first improved values of the initial solution vector $\{\delta_1\}$ as determined by these methods. Denoting the respective corrections as $\{\delta'_2\}$ and $\{\delta''_2\}$, we have, from equations (19)–(21),

$$\{\delta''_2\} = \{\delta_1\} + [A]^{-1}[P_2]\{\delta_1\} \tag{29}$$

for the effective force method, and

$$\{\delta'_2\} = \{\delta_1\} + [A]^{-1}[P_2]\{\delta'_2\} \tag{30}$$

for the matrix iteration method, with $[A] - [P_2] \equiv [A_2^*]$. Comparing these equations and taking into consideration the properties of $[A]$ and $[P]$ and the condition $\{\delta'_2\}^T\{\delta_0\} > \{\delta_1\}^T\{\delta_0\}$ (from the decrease in the energy approximation), it seems reasonable to conclude that equation (30) affords a greater improvement in the solution vector. Consequently,

at the next step the elements of $[P_3(\delta_2'')]$ will be less than the elements of $[P_3(\delta_2')]$, and as the P_3'' operator acts upon the to-be-computed vector $\{\delta_3'\}$, the matrix iteration method again will yield the more improved solution. Continuation of this argument leads to the conclusion that the iterative process of equations (21) will require fewer cycles to satisfy a specified convergence criterion than will the process of equations (19), for monotonically increasing $\psi(x, y)$. Numerical confirmation is presented in Section 6. A conceptual comparison of the two methods is depicted by Fig. 2.

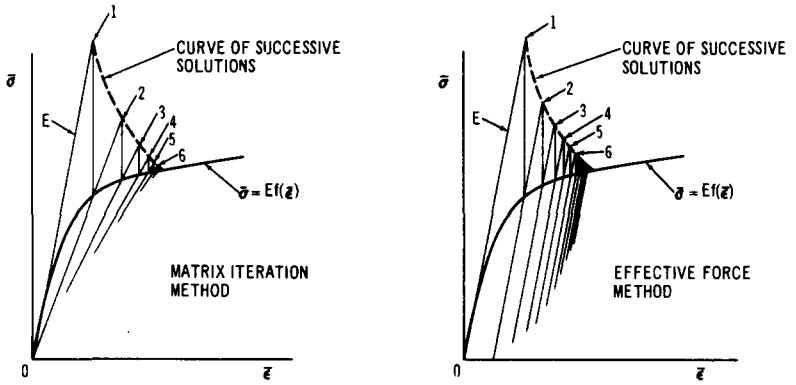

FIG. 2. Comparison of successive solution methods.

An additional consideration that applies equally to the two discretizations of the quasi-linear field equations is the question of general accuracy of the variational approach at and near the boundary. An investigation of the discretization (or truncation) errors associated with energy-derived difference equations in thermoelastic stress problems given in [19] can readily be extended to the plane strain plasticity problem treated herein. The principal result is that a difference equation at the boundary can be represented as a linear combination of the corresponding natural boundary condition and quasi-linear field equation, converging to the discretized boundary condition with decreasing grid size. The boundary discretization errors primarily depend upon the grid density normal to the boundary. Hence, effective error control is obtainable with only local changes in network spacing, as demonstrated numerically in [19]. For a uniform grid and boundary segment normal to the X-axis, denoting BC_x as the natural boundary condition, FE_x as the field equation, and DE_x as the difference equation:

$$(DE_x)_{ij} = (BC_x)_{ij} - \frac{h}{2}(FE_x)_{ij} + O(h^2) \tag{31}$$

or, simply

$$(BC_x)_{ij} - (DE_x)_{ij} = O(h). \tag{32}$$

In terms of the discretized boundary condition and field equation (using second order approximations for the derivatives), the difference equation may be expressed

$$(DE_x)_{ij} = (\overline{BC}_x)_{ij} - \frac{h}{2}(\overline{FE}_x)_{ij} \tag{33}$$

which converges to the discretized boundary condition

$$(A^*\bar{u}_{,x})_{ij}+(\lambda^*\bar{v}_{,y})_{ij}-T_{ij}=X_{ij} \tag{34}$$

as h approaches zero, where T_{ij} replaces $(3\lambda+2G)\alpha T$.

In contrast to the energy-derived equations, the truncation errors at the boundary from a direct finite-difference discretization of the differential equations generally are of $O(h^2)$. Such is the result, for example, if the method of quadratic approximation suggested by Greenspan [20] and the writer [21, 22] is used. However, for this method of formulation as well as for the various boundary approximation techniques which involve fictitious points (see [23] and [24]), the symmetry of the material-stiffness matrix is destroyed and it becomes extremely difficult if not impossible to establish positive definiteness. In the author's opinion, the advantages of having these properties far outweigh the disadvantage of the somewhat larger discretization errors at the boundary obtained from the energy formulation.

It is important to note that the convergence arguments and analytical comparisons of the effective force and matrix iteration methods given herein depend only upon the general characteristics of the matrices $[A]$ and $[P]$ and not upon the details of the quadrature formulas used in the integral approximations. Thus, the discussions in this and the following section apply equally well to a triangulated net with pyramid functions used to approximate the displacement field [18, 25, 26]—that is, to the basic element of the finite element displacement method [27]. In addition, the energy formulation of the difference equations for such elements leads to the same $O(h)$ truncation errors at the boundary [26].

5. GENERAL REMARKS ON CONVERGED SOLUTIONS

As briefly discussed, the truncation errors of the finite-difference discretization are well-defined for a given plastic strain field and are identical for the numerical models presented. In an actual iterative computational process, however, these errors may lead to somewhat different final results as determined by the two successive solution methods. This because the plasticity parameters ω and ψ are obtained at each network point and in each cycle by numerically differentiating the most recently computed displacement fields, and these fields differ after the initial "elastic" solution $\{\delta_1\}$. In addition, at each iteration step there are round-off errors associated with the solution of either of equations (19) or (21) which may be of consequence, depending upon the size of problem and the solution algorithm adopted.

Consider the nth solution step of equations (21). Since the matrix $[A^*]$ is positive definite and block tridiagonal (as discussed in [19]), it can be decomposed into the product of lower and upper triangular matrices $[L_m]$ and $[U_m]$:

$$[A^*(\psi_m)]\{\delta_m\}=[L_m][U_m]\{\delta_m\}=\{\delta_0\}. \tag{35}$$

The structure of $[A^*]$ is preserved upon triangular decomposition, from which $[L_m]$ and $[U_m]$ are block bi-diagonal. In terms of submatrices we may write

$$[A_m^*]\equiv\begin{bmatrix}C_{i-1}^T A_i C_i\end{bmatrix}=[L_m][U_m]\equiv\begin{bmatrix}L_i I & 0\end{bmatrix}\begin{bmatrix}0 & D_i U_i\end{bmatrix} \tag{36}$$

where the order of submatrix A_i equals the number of unknown displacements in row i of the finite-difference grid. The computations required in the decomposition are given by

the recursive formulas

$$\begin{aligned} D_1 &= A_1, \qquad U_i = C_i \\ L_i &= C_{i-1}^T D_{i-1}^{-1}, \qquad i > 1 \\ D_i &= A_i - L_i C_{i-1}, \qquad i > 1. \end{aligned} \tag{37}$$

The solution vector is obtained by the forward–backward substitution process

$$\begin{aligned} [L_m]\{Y_m\} &= \{\delta_0\} \\ [U_m]\{\delta_m\} &= \{Y_m\}. \end{aligned} \tag{38}$$

The rounding errors inherent in this solution algorithm are primarily associated with the decomposition process of equation (36) [28]. They have not proved significant in any of the problems investigated (see Section 6) and henceforth will be considered secondary to the truncation errors of the successive solutions method. Turning attention to these latter errors and denoting by σ_m^* and ε_m^* the equivalent stress and strain actually computed from the displacement vector $\{\delta_m\}$, numerical convergence of the recursion equations (21) after m steps implies that

$$\left| \frac{E_m^*}{E}\varepsilon_m^* - f(\varepsilon_m^*) \right| \leq e_1 \tag{39}$$

at each network point, where e_1 is a specified criterion of "error" tolerance. The variables in this equation are computed from the relationships

$$\begin{aligned} E_m^* &= \frac{E}{1+\psi_m}, \qquad \nu_m^* = \frac{\nu + 0{\cdot}5\psi_m}{1+\psi_m}, \qquad \psi_m = \frac{\varepsilon_{m-1}^*}{f(\varepsilon_{m-1}^*)} - 1 \\ \varepsilon_m^* &= \frac{1}{1+\nu_m^*}\sqrt{[\{(\bar{u}_{,x}^2 - \bar{u}_{,x}\bar{v}_{,y} + \bar{v}_{,y}^2) + \tfrac{3}{4}(\bar{u}_{,y} + \bar{v}_{,x})^2\}_{m-1}]}. \end{aligned} \tag{40}$$

Thus, equation (39) states that σ_m^* and ε_m^*, which identically satisfy the equation $\sigma_m^* = E_m^*\varepsilon_m^*$, also satisfy the tensile stress–strain law (Fig. 2) $\bar{\sigma} = Ef(\bar{\varepsilon})$ to within the specified tolerance. If e_1 be taken equal to zero over the computational digits of interest, the convergence criterion becomes $\psi_{m+1} = \psi_m$. Denoting $\Delta\bar{\varepsilon}_m$ as the truncation error in the computed value ε_m^* of the "true" equivalent strain $\bar{\varepsilon}_m$ (such that $\varepsilon_m^* = \bar{\varepsilon}_m + \Delta\bar{\varepsilon}_m$), and restricting attention to a linear strain-hardening material of tangent modulus E_T

$$f(\bar{\varepsilon}) = (1-t)\varepsilon_0 + t\bar{\varepsilon} \tag{41}$$

where $t = E_T/E \ll 1$, we find that, from equations (40) and (41),

$$\psi_{m+1} + 1 = \frac{\varepsilon_m^*}{f(\varepsilon_m^*)} = \frac{\bar{\varepsilon}_m + \Delta\bar{\varepsilon}_m}{f(\bar{\varepsilon}_m) + t\Delta\bar{\varepsilon}_m} \doteq \frac{\bar{\varepsilon}_m}{f(\bar{\varepsilon}_m)} + \left[1 - t\frac{\bar{\varepsilon}_m}{f(\bar{\varepsilon}_m)}\right]\frac{\Delta\bar{\varepsilon}_m}{f(\bar{\varepsilon}_m)}. \tag{42}$$

Hence, defining $p = \bar{\varepsilon}_m/\bar{\varepsilon}_0$, a measure of the error in the "converged" value of ψ is

$$|e_\psi| = \left| \frac{\bar{\varepsilon}_m}{f(\bar{\varepsilon}_m)} - \frac{E}{E_m^*} \right| = \frac{(1-t)|\Delta\bar{\varepsilon}_m|}{[1+2t(p-1)]\varepsilon_0} < \frac{|\Delta\bar{\varepsilon}_m|}{\varepsilon_0} \tag{43}$$

which is of order h or higher. There is no apparent way of eliminating this error from the computational process other than to use a finer grid, which in turn may lead to round-off

errors of greater significance. The combination of the two sources of error does not appear to be critical, however, based upon numerical results for problems involving relatively large (1000 to 2000 order) matrices. These results are reported in part in the following Section.

6. NUMERICAL APPLICATION

The methods of discretization and successive solution discussed herein have been programmed for the analysis of an infinitely long bar with a parabolic temperature variation in the transverse Y-direction (Fig. 3). The finite-difference grids that have been

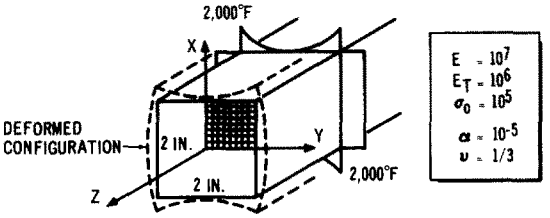

FIG. 3. Longitudinally constrained, thermally loaded bar.

used in the numerical computations, with each corresponding matrix order in parentheses, are (one quadrant of the bar only)

(a) Uniform 21×21 grid ($N = 840$)
(b) Quasi-uniform 22×22 grid ($N = 924$)
(c) Uniform 30×30 grid ($N = 1740$).

The quasi-uniform grid is obtained from the uniform 21×21 grid by the addition of a grid line halfway between the free boundary and the first interior grid line.

The programs have been written in FORTRAN IV using single precision arithmetic, and all computer times quoted are for a UNIVAC 1108 digital computer with intermediate access drum storage. Two studies have been made to assess rounding errors in the matrix decomposition and solution process: (1) comparative solutions of 90° rotated temperature fields, as reported in [19] for the analogous plane stress problem; and (2) use of a double precision routine to improve the solution vector via an iterative technique given by Wilkinson [28]. These studies indicate that, even for the 1740 order matrix, the initial solution vector has errors only in the sixth digit. For this reason, single precision arithmetic has been deemed adequate, and Wilkinson's technique of improving a solution vector has not been used in the computations of the elastic–plastic problem.

With the objective of attaining sufficiently accurate solutions (from the standpoint of design stress analysis) without undue expenditure of computer time, a convergence criterion has been chosen of less than one hundredths of one per cent change in the vector norm $\|\delta\|$, where $\|\delta\|^2 = \sum_{i=1}^{N} \delta_i^2$. Its adequacy has been tested by comparing results for the 21×21 grid based upon this criterion with those obtained by continuing the successive solutions process until the vector norm ceases to change through eight places. From this

it is concluded that the 0·01 percent criterion will yield displacements converged to five significant figures and stresses converged to four. All results cited herein are based upon the use of this criterion.

A comparison of the matrix iteration and effective force methods as to number of iterations for convergence and total running time for the entire solution is given in Table 1.

TABLE 1. COMPARISON OF CYCLES TO CONVERGENCE AND COMPUTING TIMES

	Elastic solution		Matrix iteration		Effective force	
	$N = 924$	$N = 1740$	$N = 924$	$N = 1740$	$N = 924$	$N = 1740$
Iterations	1	1	5	5	9	10
Computer time (min)	1·2	3·0	5·5	14·7	5·4	14·6

The numbers shown correspond to a maximum temperature of 2000°F. Other problems with higher and lower temperatures also were calculated. In all cases, the matrix iteration method converged in significantly fewer cycles than did the effective force method, confirming the arguments of Section 4. Relative computational time is another matter, however, and it is important to comment upon this as it pertains to the larger question of developing efficient plastic strain analysis programs for use in thermostructural design studies. If the system of simultaneous equations were to be solved by an iterative method (such as successive overelaxation) at each step, the computational times would be precisely the same per cycle by either method. In that instance the matrix iteration procedure clearly would be superior due to the fewer number of cycles required for convergence. But when an efficient tridiagonal matrix decomposition technique is available, as here, the per cycle time of the effective force method can be considerably less. This is because the submatrices L_i, D_i, U_i of the decomposed matrix $[A]$ can be saved in intermediate storage once they are computed for the initial solution $\{\delta_1\}$, and only the forward–backward substitution process of equations (38) need be repeated in successive cycles. In contrast, at each step of the matrix iteration method, $[A_m^*]$ first must be decomposed before the substitution process can be carried-out, and this decomposition requires the greater part of the solution time. Thus, although the total running times are comparable for the problems investigated (Table 1), given a matrix inversion algorithm, the effective force method could prove to be the more efficient for significantly larger order problems (say, $N > 4000$). It also should be noted that the effective force concept is more readily adapted for use with incremental theories of plasticity [10, 15, 29]. (For comparisons, in incremental loading problems, of the effective force method and another technique variously called the tangent stiffness [15] or tangent modulus [30, 31] method, see [15] and [32].)

In Section 5, the error sources which can lead to differences between converged solutions corresponding to the two iterative methods were discussed. In Table 2 and Figs. 4, 5, and 6, the significance of these errors can be assessed through a comparison of results for the 30×30 grid. Table 2 gives the u-, and v-displacements along the free edges $x = 1$ and $y = 1$, respectively, as computed by each method. From a comparison of values over all the network points, the results in general differ by one per cent or less in the u displacements and two per cent or less in the v displacements. In Fig. 4, the σ_x stresses along representative grid lines obtained from the two plastic solutions are plotted, together with the values

TABLE 2. COMPARISON OF DISPLACEMENTS ALONG FREE EDGES

	Matrix iteration	Effective force	Matrix iteration	Effective force
y, x	u displacements ($x = 1$)		v displacements ($y = 1$)	
0	0·00204	0·00204	0·01159	0·01165
2/29	0·00215	0·00215	0·01157	0·01163
4/29	0·00247	0·00247	0·01152	0·01158
6/29	0·00300	0·00300	0·01143	0·01148
8/29	0·00374	0·00374	0·01130	0·01135
10/29	0·00468	0·00467	0·01111	0·01116
12/29	0·00579	0·00578	0·01087	0·01092
14/29	0·00708	0·00706	0·01055	0·01060
16/29	0·00851	0·00849	0·01014	0·01019
18/29	0·01008	0·01005	0·00961	0·00966
20/29	0·01178	0·01174	0·00894	0·00899
22/29	0·01361	0·01357	0·00810	0·00815
24/29	0·01555	0·01551	0·00704	0·00709
26/29	0·01753	0·01750	0·00572	0·00577
28/29	0·01952	0·01949	0·00408	0·00413
1	0·02053	0·02051	0·00313	0·00318

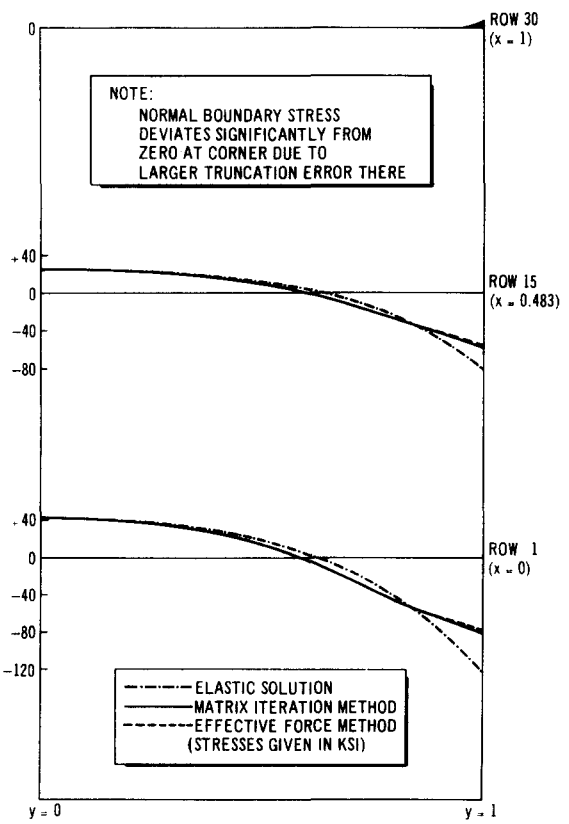

FIG. 4. Comparison of transverse σ_x stresses.

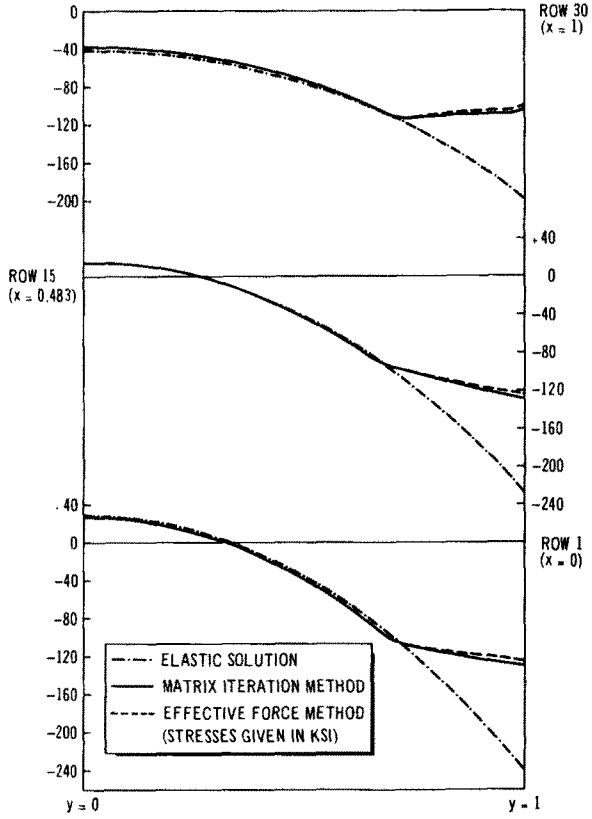

FIG. 5. Comparison of longitudinal σ_z stresses.

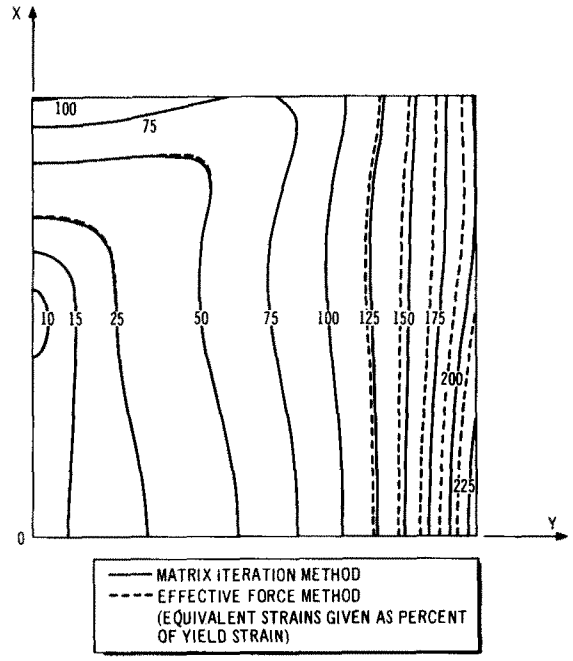

FIG. 6. Comparison of equivalent strains.

corresponding to the "elastic" solution. Similar plots are given in Fig. 5 for the longitudinal σ_z stress. In Fig. 6, contour lines of equal equivalent strain $\bar{\varepsilon}$ are shown. Although the observed differences between numerical results are not significant from the view of applying these methods in practical stress analysis problems, it would be desirable to know which of the two computational processes is the more accurate. This determination might be realized through a comprehensive error analysis or through comparison with a closed form solution, but neither the error analysis nor the "exact" solution is readily accomplished.

Lastly, the monotonic convergence to the minimum potential energy from above is shown in Table 3, together with a comparison of the convergence of the vector norms corresponding to the two solution methods.

TABLE 3. POTENTIAL ENERGY AND VECTOR NORMS

Cycle	1	2	3	4	5	9	10
$\bar{\Pi}_m - D_T\left\{\begin{matrix}\text{Matrix}\\ \text{iter.}\end{matrix}\right\}$	−0·000265	−0·007493	−0·007532	−0·007537⁻	−0·007537	—	—
$\|\delta_m\|\left\{\begin{matrix}\text{Matrix}\\ \text{iter.}\end{matrix}\right\}$	0·20906	0·20717	0·20672	0·20665⁺	0·20665⁻	—	—
$\|\delta_m\|\left\{\begin{matrix}\text{Effec.}\\ \text{force}\end{matrix}\right\}$	0·20906	0·20898	0·20753	0·20684	0·20654	0·20632⁺	0·20632

7. SUMMARY

An analytical and numerical comparison between two methods of successive solution in elasto-plastic plane strain problems has been presented. Minimization of the discretized potential energy function of a Hencky–Nadai material has been shown to yield a finite-difference displacement formulation of the effective force method that is compatible with the numerical model of the material-stiffness matrix method given in a previous paper. A sufficient condition for monotonic convergence to the minimum energy from above has been established, and convergence of the matrix iteration method in fewer cycles than required for the force iteration method has been demonstrated. In contrast, the effective force method requires equal if not less computer time than matrix iteration when a matrix decomposition algorithm is available. Direct solutions of the quasi-tridiagonal, positive definite stiffness matrices by block decomposition techniques have been shown to yield accurate and efficient determinations of stress and displacement fields by either general method.

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APPENDIX

For a rectangular element ij, $i+1j$, $i+1j+1$, $ij+1$ (Fig. 1) appropriate formulas for evaluation of the terms in equation (13) are

$$\iint_{\Delta R} G\bar{u}_{,x}^2 \, dx \, dy = \tfrac{1}{2}G\{(u_{i+1j}-u_{ij})^2+(u_{i+1j+1}-u_{ij+1})^2\}\frac{\eta_j}{\xi_i}$$

$$\begin{aligned}
\iint_{\Delta R} \tfrac{1}{2}G(\bar{u}_{,y}+\bar{v}_{,x})^2\,dx\,dy &= \tfrac{1}{8}G\left\{\left(\frac{u_{ij+1}-u_{ij}}{\eta_j}+\frac{v_{i+1j}-v_{ij}}{\xi_i}\right)^2+\left(\frac{u_{i+1j+1}-u_{i+1j}}{\eta_j}\right.\right.\\
&\quad\left.+\frac{v_{i+1j}-v_{ij}}{\xi_i}\right)^2+\left(\frac{u_{ij+1}-u_{ij}}{\eta_j}+\frac{v_{i+1j+1}-v_{ij+1}}{\xi_i}\right)^2\\
&\quad\left.+\left(\frac{u_{i+1j+1}+u_{i+1j}}{\eta_j}+\frac{v_{i+1j+1}-v_{ij+1}}{\xi_i}\right)^2\right\}\xi_i\eta_j\\
\iint_{\Delta R} G\bar{v}_{,y}^2\,dx\,dy &= \tfrac{1}{2}G\{(v_{ij+1}-v_{ij})^2+(v_{i+1j+1}-v_{i+1j})^2\}\frac{\xi_i}{\eta_j}\\
\iint_{\Delta R} \tfrac{1}{2}\lambda(\bar{u}_{,x}+\bar{v}_{,y})^2\,dx\,dy &= \frac{1}{8}\lambda\left\{\left(\frac{u_{i+1j}-u_{ij}}{\xi_i}+\frac{v_{ij+1}-v_{ij}}{\eta_j}\right)^2\right.\\
&\quad+\left(\frac{u_{i+1j}-u_{ij}}{\xi_i}+\frac{v_{i+1j+1}-v_{i+1j}}{\eta_j}\right)^2+\left(\frac{u_{i+1j+1}-u_{ij+1}}{\xi_i}+\frac{v_{ij+1}-v_{ij}}{\eta_j}\right)^2\\
&\quad\left.+\left(\frac{u_{i+1j+1}-u_{ij+1}}{\xi_i}+\frac{v_{i+1j+1}-v_{i+1j}}{\eta_j}\right)^2\right\}\xi_i\eta_j.
\end{aligned} \tag{A1}$$

Corresponding formulas for a triangular element ij, $i+1j$, $ij+1$ are

$$\begin{aligned}
\iint_{\Delta R} G\bar{u}_{,x}^2\,dx\,dy &= \tfrac{1}{2}G(u_{i+1j}-u_{ij})^2\frac{\eta_j}{\xi_i}\\
\iint_{\Delta R} \tfrac{1}{2}G(\bar{u}_{,y}+\bar{v}_{,x})^2\,dx\,dy &= \frac{1}{4}G\left(\frac{u_{ij+1}-u_{ij}}{\eta_j}+\frac{v_{i+1j}-v_{ij}}{\xi_i}\right)^2\xi_i\eta_j\\
\iint_{\Delta R} G\bar{v}_{,y}^2\,dx\,dy &= \tfrac{1}{2}G(v_{ij+1}-v_{ij})^2\frac{\xi_i}{\eta_j}\\
\iint_{\Delta R} \tfrac{1}{2}\lambda(\bar{u}_{,x}+\bar{v}_{,y})^2\,dx\,dy &= \frac{1}{4}\lambda\left(\frac{u_{i+1j}-u_{ij}}{\xi_i}+\frac{v_{ij+1}-v_{ij}}{\eta_j}\right)^2\xi_i\eta_j.
\end{aligned} \tag{A2}$$

The integral approximations of equations (A1) and (A2) are non-negative and equal to zero only if the following relations hold for all i,j:

$$\begin{aligned}
u_{i+1j} &= u_{ij}, \qquad v_{ij+1} = v_{ij}\\
(u_{ij+1}-u_{ij})+\frac{\eta_j}{\xi_i}(v_{i+1j}-v_{ij}) &= 0.
\end{aligned} \tag{A3}$$

These equations describe a rigid body translation of arbitrary magnitude combined with a vanishingly small rigid body rotation. Hence, the discretization is "consistent" with the non-uniqueness of the continuum solution of the first boundary value problem. To eliminate the non-uniqueness in a particular numerical problem, it is only necessary to specify one u, one v, and at least one additional u or v, in which case the matrix $[A]$ of the mixed boundary value problem is positive definite, as shown in [7] for the material-stiffness matrix of the matrix iteration method.

To ensure compatibility of the numerical model of the effective force method with that of the matrix iteration method given in [7], similar quadrature formulas are used for

evaluation of the terms in equation (14). For the rectangular element,

$$\begin{aligned}
\iint_{\Delta R} \tfrac{2}{3}G\omega\bar{u}_{,x}^2\,dx\,dy &= \tfrac{1}{6}G\{(\omega_{i+1j}+\omega_{ij})(u_{i+1j}-u_{ij})^2 \\
&\quad +(\omega_{i+1j+1}+\omega_{ij+1})\cdot(u_{i+1j+1}-u_{ij+1})^2\}\frac{\eta_j}{\xi_i} \\
\iint_{\Delta R} \tfrac{2}{3}G\omega\bar{u}_{,x}\bar{v}_{,y}\,dx\,dy &= \tfrac{1}{6}G\{\omega_{ij}(u_{i+1j}-u_{ij})(v_{ij+1}-v_{ij})+\omega_{i+1j}(u_{i+1j}-u_{ij}) \\
&\quad \cdot(v_{i+1j+1}-v_{i+1j})+\omega_{ij+1}(u_{i+1j+1}-u_{ij+1})(v_{ij+1}-v_{ij}) \\
&\quad +\omega_{i+1j+1}(u_{i+1j+1}-u_{ij+1})(v_{i+1j+1}-v_{i+1j})\} \\
\iint_{\Delta R} \tfrac{2}{3}G\omega\bar{v}_{,y}^2\,dx\,dy &= \tfrac{1}{6}G\{(\omega_{ij+1}+\omega_{ij})(v_{ij+1}-v_{ij})^2+(\omega_{i+1j+1}+\omega_{i+1j}) \\
&\quad \cdot(v_{i+1j+1}-v_{i+1j})^2\}\frac{\xi_i}{\eta_j} \\
\iint_{\Delta R} \tfrac{1}{2}G\omega(\bar{u}_{,y}+\bar{v}_{,x})^2\,dx\,dy &= \frac{1}{8}G\left\{\omega_{ij}\left(\frac{u_{ij+1}-u_{ij}}{\eta_j}+\frac{v_{i+1j}-v_{ij}}{\xi_i}\right)^2\right. \\
&\quad +\omega_{i+1j}\left(\frac{u_{i+1j+1}-u_{i+1j}}{\eta_j}+\frac{v_{i+1j}-v_{ij}}{\xi_i}\right)^2 \\
&\quad +\omega_{ij+1}\left(\frac{u_{ij+1}-u_{ij}}{\eta_j}+\frac{v_{i+1j+1}-v_{ij+1}}{\xi_i}\right)^2 \\
&\quad \left.+\omega_{i+1j+1}\left(\frac{u_{i+1j+1}-u_{i+1j}}{\eta_j}+\frac{v_{i+1j+1}-v_{ij+1}}{\xi_i}\right)^2\right\}\xi_i\eta_j
\end{aligned} \tag{A4}$$

with corresponding equations for the triangular element.

Summing the quadrature formulas over the network elements surrounding point ij and then differentiating with respect to u_{ij}, the difference equation in the X-direction is obtained:

$$\begin{aligned}
&\left[2(\eta_j+\eta_{j-1})\left(\frac{1}{\xi_i}+\frac{1}{\xi_{i-1}}\right)(\lambda+2G)+2(\xi_i+\xi_{i-1})\left(\frac{1}{\eta_j}+\frac{1}{\eta_{j-1}}\right)(G)\right]u_{ij}-2\left(\frac{\eta_j+\eta_{j-1}}{\xi_i}\right) \\
&\quad\cdot(\lambda+2G)u_{i+1j}-2\left(\frac{\eta_j+\eta_{j-1}}{\xi_{i-1}}\right)(\lambda+2G)u_{i-1j}-2\left(\frac{\xi_i+\xi_{i-1}}{\eta_j}\right)(G)u_{ij+1} \\
&\quad -2\left(\frac{\xi_i+\xi_{i-1}}{\eta_{j-1}}\right)(G)u_{ij-1}-(\lambda+G)(v_{i+1j+1}-v_{i+1j-1}-v_{i-1j+1}+v_{i-1j-1})+(\eta_j+\eta_{j-1}) \\
&\quad\cdot(T_{i+1j}-T_{i-1j})h-(\xi_i+\xi_{i-1})(\eta_j+\eta_{j-1})(F_x)_{ij}h^2 \\
&\quad =\left[\frac{4}{3}(\eta_j+\eta_{j-1})G\left\{\frac{\omega_{i+1j}+\omega_{ij}}{\xi_i}+\frac{\omega_{i-1j}+\omega_{ij}}{\xi_{i-1}}\right\}+(\xi_i+\xi_{i-1})G\left\{\frac{\omega_{ij+1}+\omega_{ij}}{\eta_j}+\frac{\omega_{ij-1}+\omega_{ij}}{\eta_{j-1}}\right\}\right] \\
&\qquad\times u_{ij}-\frac{4}{3}\left(\frac{\eta_j+\eta_{j-1}}{\xi_i}\right)G(\omega_{i+1j}+\omega_{ij})u_{i+1j} \\
&\qquad -\frac{4}{3}\left(\frac{\eta_j+\eta_{j-1}}{\xi_{i-1}}\right)G(\omega_{i-1j}+\omega_{ij})u_{i-1j}-\left(\frac{\xi_i+\xi_{i-1}}{\eta_j}\right)G(\omega_{ij+1}+\omega_{ij})u_{ij+1}
\end{aligned}$$

$$-\left(\frac{\xi_i+\xi_{i-1}}{\eta_{j-1}}\right)G(\omega_{ij-1}+\omega_{ij})u_{ij-1}-G(\omega_{ij+1}-\tfrac{2}{3}\omega_{i+1j})v_{i+1j+1}$$
$$+G(\omega_{ij-1}-\tfrac{2}{3}\omega_{i+1j})v_{i+1j-1}+G(\omega_{ij+1}-\tfrac{2}{3}\omega_{i-1j})v_{i-1j+1}$$
$$-G(\omega_{ij-1}-\tfrac{2}{3}\omega_{i-1j})v_{i-1j-1}. \tag{A5}$$

The difference equation in the Y-direction may be written by interchanging u and v, x and y, ξ and η, and i and j. Difference equations for points on or adjacent to the boundary or developed in similar fashion.

For the matrix iteration method corresponding to equations (8), the quasi-linear finite-difference equations are derived in [7] in terms of the pseudo material properties λ^*, G^*. The equation in the X-direction at a typical interior point is (denoting $A^* = \lambda^* + 2G^*$)

$$\left[\frac{\eta_j+\eta_{j-1}}{\xi_i}(A^*_{i+1j}+A^*_{ij})+\frac{\eta_j+\eta_{j-1}}{\xi_{i-1}}(A^*_{i-ij}+A^*_{ij})+\frac{\xi_i+\xi_{i-1}}{\eta_j}(G^*_{ij+1}+G^*_{ij})+\frac{\xi_i+\xi_{i-1}}{\eta_{j-1}}.(G^*_{ij-1}+G^*_{ij})\right]$$
$$\times u_{ij}-\frac{\eta_j+\eta_{j-1}}{\xi_i}(A^*_{i+1j}+A^*_{ij})u_{i+1j}-\frac{\eta_j+\eta_{j-1}}{\xi_{i-1}}(A^*_{i-1j}+A^*_{ij})u_{i-1j}$$
$$-\frac{\xi_i+\xi_{i-1}}{\eta_j}(G^*_{ij+1}+G^*_{ij})u_{ij+1}-\frac{\xi_i+\xi_{i-1}}{\eta_{j-1}}(G^*_{ij-1}+G^*_{ij})u_{ij-1}-(\lambda^*_{i+1j}+G^*_{ij+1})v_{i+1j+1}$$
$$+(\lambda^*_{i+1j}+G^*_{ij-1})v_{i+1j-1}+(\lambda^*_{i-1j}+G^*_{ij+1})v_{i-1j+1}-(\lambda^*_{i-1j}+G^*_{ij-1})v_{i-1j-1}$$
$$+(\eta_j+\eta_{j-1})(T_{i+1j}-T_{i-1j})h-(\xi_i+\xi_{i-1})(\eta_j+\eta_{j-1})(F_x)_{ij}h^2=0. \tag{A6}$$

From equations (7), (9) and (10):

$$\begin{aligned}(\lambda+2G)-\tfrac{4}{3}G\omega_{ij} &= A^*_{ij}\\ G-G\omega_{ij} &= G^*_{ij}\\ \lambda+\tfrac{2}{3}G\omega_{ij} &= \lambda^*_{ij}.\end{aligned} \tag{A7}$$

Thus, equation (A5) can be transformed identically into equation (A6) for a given plastic strain field.

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Абстракт—Исследуется две общие формулировки и методы решения задач лпоского деформационного состояния пластичности: Метод зффективной силы и метод итерации матрицы козффициентов жесткости материала. Выводится согласные, конечно-разностные молели путем минимализации надискретизации функции потенциальной знергии по общей конфигурации контура, используя закон упрочнения Хенки-Надая, с цалью прелставления математически поведения материала. Суммарные матрицы козффициентов жесткости материала являются симметричными, положительно определенными и блочными, трехдиагональными. Приводятся аналитические и численные исследования характеристик точности и сходимости методов для матрицы ранга 1000.